respectively. It is seen that the thermal wave velocity and damping in a zinc crystal depend weakly on the propagation direction.

Thus, a characteristic singularity of acceleration wave propagation in an anisotropic medium is the deviation of the wave tubes from the normal vector. For $\tau = 0$ a second quasi-longitudinal wave appears which damps out more rapidly than the first. The relaxation time τ turns out to exert substantial influence on the nature of quasi-longitudinal and quasi-transverse wave propagation.

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ON DYNAMIC EFFECTS IN AN ELASTIC HALF-SPACE UNDER "THERMAL IMPACT"

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The general uncoupled dynamical problem of thermoelasticity for a half-space under the condition of a thermal impact with a finite rate of change in temperature on its boundary is solved by the method of principal (fundamental) functions within the framework of a generalized theory of heat conduction.

An elastic steel half-space is analyzed as an illustration. The problem on thermal stresses originating in an elastic half-space due to thermal impact produced by a jump change in temperature on the boundary was first analyzed in [1]. Since the temperature change on the boundary occurs at a finite rate, it is generally impossible to realize the thermal impact considered in [1] physically. The dynamic effects in an elastic half-space under a thermal impact with finite rate of change in the temperature on the boundary have been studied in [2]. For high rates of change of the heat flux we obtain a generalized wave equation of heat conduction [3] taking into account the finite velocity of heat propagation. Hence, the solution of the ordinary parabolic heat conduction equation used in [1, 2] does not correspond to the true temperature field. The problems of [1, 2] have been examined in [4, 5], respectively, within the framework of a generalized theory of heat conduction.

1. Formulation of the problem. Let an elastic half-space $z \geqslant 0$, as well as the medium in the domain $z < 0$ be initially at the temperature $t_0 = 0$, and then let the temperature of the medium adjoining the surface of the half-space $z = 0$ grow linearly from $t_0 = 0$ and reach the finite value α_0 within a small, but nonzero, time interval τ_0. For $\tau > 0$ convective heat exchange according to Newton law occurs between the surface and the medium.

To determine the temperature field in this case it is required to find a bounded, sufficiently smooth solution of the problem

$$\frac{1}{w_r^2}\frac{\partial^2 t}{\partial \tau^2} + \frac{\bar{c}\gamma}{\lambda}\frac{\partial t}{\partial \tau} = \frac{\partial^2 t}{\partial z^2}, \quad t|_{\tau=0} = 0, \quad \frac{\partial t}{\partial \tau}\Big|_{\tau=0} = 0 \tag{1.1}$$

$$\left[\frac{\partial t}{\partial z} - h\left(1 + \tau_r \frac{\partial}{\partial \tau}\right)(t - t_c)\right]\Big|_{z=0} = 0, \quad \lim_{z\to\infty} t(\tau, z) = 0$$

$$t_c|_{z=0} = \varphi(z) = \begin{cases} 0, & \tau_0 \leqslant 0 \\ \alpha_0 \tau / \tau_0 & 0 \leqslant \tau \leqslant \tau_0 \\ \alpha_0 & \tau_0 \leqslant \tau \end{cases}$$

$$\tau = \alpha_0 \frac{\tau}{\tau_0} J_-(\tau_0 - \tau) J_-(\tau) + \alpha_0 J_-(\tau - \tau_0)$$

Here $J_-(s)$ is an asymmetric unit Heaviside function [6]. If it is now considered that the half-space was initially stress-free and that there are not stresses on its surface $z = 0$ during heating, then the problem [2]

$$\frac{\partial^2 \sigma_z}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \sigma_z}{\partial \tau^2} = a \frac{\partial^2 t}{\partial \tau^2}, \quad a = \frac{1+\mu}{1-\mu}\gamma \alpha_\tau \tag{1.2}$$

$$\sigma_z(\tau, z)|_{\tau=0} = 0, \quad \frac{\partial \sigma_z}{\partial \tau}\Big|_{\tau=0} = 0, \quad \sigma_z|_{z=0} = 0, \quad \sigma_z|_{z=\infty} = 0$$

must be solved to determine the stresses. Here τ_r is the relaxation time of the thermal process, w_r is as yet a large, but finite velocity of heat propagation, λ is the coefficient of heat conduction, c is the specific heat of the substance, γ is the density of the substance, h is the relative coefficient of heat exchange, α_τ is the coefficient of linear expansion of the material, a is the speed of sound, and μ is a Lamé constant.

2. Mixed problem for the wave equation. Let us consider the problem of finding a sufficiently smooth solution bounded at infinity for the problem

$$L[u] \equiv b_0^2 \frac{\partial^2 u}{\partial \tau^2} + b_1^2 \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial z^2} = f(\tau, z) \tag{2.1}$$

$$u|_{\tau=0} = \varphi_1(z), \quad \frac{\partial u}{\partial \tau}\Big|_{\tau=0} = \varphi_2(z)$$

$$B[u]|_{z=0} \equiv \left[\frac{\partial}{\partial z} - \alpha h\left(1 + \frac{\tau_r}{\beta}\frac{\partial}{\partial \tau}\right)\right] u\Big|_{z=0} = -h\left(1 + \frac{\alpha}{\beta}\tau_r\frac{\partial}{\partial \tau}\right) u_c \equiv -h\psi(\tau)$$

$$\lim_{z\to\infty} u(\tau, z) = 0$$

in the domain

$$\Pi^+ = \{(z, \tau);\quad 0 \leqslant z < \infty,\quad 0 \leqslant \tau \leqslant T\,(T \leqslant \infty)\} \equiv [0, \infty) \times [0, T$$

Definitions. (1). We call the function $K(\tau, z, \xi)$ satisfying the equation $L[u] = 0$ and the conditions

$$K|_{\tau=0} = 0, \quad \frac{\partial K}{\partial \tau}\Big|_{\tau=0} = \delta_\xi, \quad B[K]|_{z=0} = 0$$

the Cauchy function of the mixed problem (2.1).

(2). We call the function $W(\tau, s, z)$ satisfying the equation $L[u] = 0$, zero initial conditions, and the boundary condition

$$B\,[W]_{z=0} = \delta_s$$

the Green's function of the mixed problem (2.1).

(3). We call the function $E(z, \xi, \tau, s)$ satisfying the equation

$$L[E] = \delta(z - \xi, \tau - s) = \delta(z - \xi) \otimes \delta(\tau - s) = \delta_\xi \otimes \delta_s$$

zero initial and boundary conditions the fundamental function of the mixed problem (2.1). Here δ_a denotes the Dirac measure concentrated at the point a and $\otimes$ is the tensor product of the generalized functions.

Following [7], it can be verified that the functions K, W and E are

$$K(\tau, z, \xi) = b_0^2\Big[\Phi(\tau, |z - \xi|) - \Phi(\tau, z + \xi) - 2\int_0^\infty \exp(-h_1 y)\frac{\partial}{\partial \xi}\Phi(\tau - \beta_1 y, z + \xi + y)\,dy\Big]$$

$$W(\tau, s, z) = 2\int_0^\infty \exp(-h_1 y)\frac{\partial}{\partial z}\Phi(\tau - s - \beta_1 y, z + y)\,dy$$

$$E(z, \xi; \tau, s) = \begin{cases} 0, & \tau \leqslant s \\ b_0^{-2}K(\tau - s; z, \xi), & \tau > s \end{cases}$$

$$k_1 = \frac{b_1^2}{2b_0^2}, \quad h_1 = \alpha h, \quad \beta_1 = \beta^{-1}\alpha h\tau_r, \quad \Phi(\tau, z) = b_0^{-2}G(\tau, z).$$

$$G(\tau, z) = \frac{1}{2}b_0\exp(-k_1\tau)\,I_0\left(k_1\sqrt{\tau^2 - b_0^2 z^2}\right)J_-(\tau - b_0 z)$$

where $G(\tau, z)$ is a fundamental solution of the Cauchy problem for the equation $L[u] = 0$. Compliance with the complement condition

$$h_1 + \beta_1 p + \sqrt{b_0^2 p^2 + b_1^2 p} \neq 0$$
$$p = p_0 + ip_1,\ p_0 > 0,\ -\infty < p_1 < +\infty$$

plays an essential part here.

Theorem. If the complement condition is satisfied, then the solution of the mixed problem (2.1) is determined by the formula

$$u(\tau, z) = \exp(-k_1\tau)\Big[\frac{b_0-\beta_1}{b_0+\beta_1}\varphi_1\Big(\frac{\tau-b_0z}{b_0}\Big)J_-(\tau-b_0z) + \tag{2.2}$$

$$\frac{1}{2}\varphi_1\Big(\frac{b_0z-\tau}{b_0}\Big)J_-(b_0z-\tau) + \frac{1}{2}\varphi_1\Big(\frac{b_0z+\tau}{b_0}\Big)\Big] +$$

$$h\int_0^\tau\int_0^\infty (z+y)\exp(-h_1y)F(\tau-s-\beta_1y, z+y)\psi(s)\,dy\,ds +$$

$$\frac{h}{b_0+\beta_1}\int_0^\tau F_1(\tau-s, z)J_-(\tau-b_0z-s)\psi(s)\,ds +$$

$$\int_0^\tau\int_0^\infty \Big[\Phi(\tau-s, |z-\xi|) - \Phi(\tau-s, z+\xi) + \frac{1}{b_0+\beta_1}\times$$

$$F_1(\tau-s, z+\xi)J_-(\tau-s-b_0(z+\xi)) + \int_0^\infty (z+\xi+y)\times$$

$$\exp(-h_1y)F(\tau-s-\beta_1y, z+\xi+y)\,dy\Big]f(\xi, s)\,d\xi\,ds + \int_0^\infty \Big[\Phi(\tau,$$

$$|z-\xi|) - \Phi(\tau, z+\xi) + \frac{1}{b_0+\beta_1}F_1(\tau, z+\xi)J_-(\tau-b_0(z+\xi)) +$$

$$\int_0^\infty \exp(-h_1y)(z+\xi+y)F(\tau-\beta_1y, z+\xi+y)\,dy\Big][b_0^2\varphi_2(\xi) +$$

$$b_1^2\varphi_1(\xi)]\,d\xi + b_0^2\int_0^\infty \Big\{k_1\Big[\Phi(\tau, z+\xi) - \Phi(\tau, |z-\xi|) + \frac{\tau}{2b_0^2}\times$$

$$[F(\tau, |z-\xi|) - F(\tau, z+\xi)] + \Big(\frac{k_1^2b_0}{2}\,\frac{\tau+\beta_1(z+\xi)}{(\beta_1+b_0)^2} -$$

$$\frac{k_1b_0+h_1}{(b_0+\beta_1)^2}\Big)F_1(\tau, z+\xi)J_-(\tau-b_0(z+\xi)) + \int_0^\infty \exp(-h_1y)\times$$

$$(z+\xi+y)\Big[\frac{2k_1^2b_0^2(\tau-\beta_1y)}{(\tau-\beta_1y)^2-b_0^2(z+\xi+y)^2}\Phi(\tau-\beta_1y, z+\xi+y) -$$

$$\Big(k_1 + 2\frac{\tau-\beta_1y}{(\tau-\beta_1y)^2-b_0^2(z+\xi+y)^2}\Big)F(\tau-\beta_1y, z+\xi+y)\Big]dy\Big\}\bar{\varphi}_1(\xi)\,d\xi$$

$$F_1(\tau, z) = \exp\Big(\frac{h_1-k_1\beta_1}{b_0+\beta_1}b_0z\Big)\exp\Big(-\frac{k_1b_0+h_1}{b_0+\beta_1}\tau\Big)$$

$$F(\tau, z) = k_1b_0\exp(-k_1\tau)\frac{I_1\big(k_1\sqrt{\tau^2-b_0^2z^2}\big)}{\sqrt{\tau^2-b_0^2z^2}}J_-(\tau-b_0z)$$

Proof. Let us rewrite (2.2) as follows:

$$u(\tau, z) = -2h\int_0^\tau\int_0^\infty \exp(-h_1y)\frac{\partial}{\partial z}\Phi(\tau-s-\beta_1y, z+y)\psi(s)\,dy\,ds +$$

$$\int_0^\tau\int_0^\infty \Big[\Phi(\tau-s, |z-s|) - \Phi(\tau-s, z+\xi) - 2\int_0^\infty \exp(-h_1y)\times$$

$$\frac{\partial}{\partial\xi}\Phi(\tau - s - \beta_1 y, z + \xi + y)\,dy\Big] f(s,\xi)\,d\xi\,ds + b_0^2\int_0^\infty \Big[\Phi(\tau, |z -$$

$$\xi|) - \Phi(\tau, z+\xi) - 2\int_0^\infty \exp(-h_1 y)\frac{\partial}{\partial\xi}\Phi(\tau - \beta_1 y, z + \xi + y)dy\Big]\times$$

$$\varphi_2(\xi)\,d\xi + \int_0^\infty \left(b_0^2\frac{\partial}{\partial\tau} + b_1^2\right)\Big[\Phi(\tau, |z-\xi|) - \Phi(\tau, z+\xi) -$$

$$2\int_0^\infty \exp(-h_1 y)\frac{\partial}{\partial\xi}\Phi(\tau - \beta_1 y, z + \xi + y)\,dy\Big]\varphi_1(\xi)\,d\xi =$$

$$W * \psi(\tau) + E ** f(\tau, z) + K * \left[\varphi_2(z) + \frac{b_1^2}{b_0^2}\varphi_1(z)\right] + \frac{\partial K}{\partial\tau} * \varphi_1(z)$$

The validity of (2.2) becomes evident if we use the properties of the functions W, K and E, theorems on the continuity and differentiability of convolutions [8] taking into account that $\left(\frac{\partial^2 K}{\partial\tau^2} * \varphi_1\right)\Big|_{\tau=0} = \left(\frac{1}{b_0^2}\frac{\partial^2}{\partial z^2} - \frac{b_1^2}{b_0^2}\frac{\partial}{\partial\tau}\right)K\Big|_{\tau=0} = -\frac{b_1^2}{b_0^2}\varphi_1$

Corollaries. (1). Formula (2.2) defines:

(a) The solution of the mixed problem (2.1) for a boundary condition of the third kind for $\alpha = 1$, $\beta = 1$;

(b) The solution of the mixed problem (2.1) for a boundary condition of the second kind for $\alpha = 0$, $h = -1$;

(c) The solution of the mixed problem (2.1) for a boundary condition of the first kind for $h \to \infty$, $\beta \to \infty$, $\alpha = 1$.

(2°). Formula (2.2) defines the solution of the mixed problem (2.1) for a pure wave equation when $b_1 = 0$, and for the parabolic equation obtained from (2.1) for $b_0 = 0$ when $\beta_1 = 0$ and $b_0 \to 0$. This latter solution has the form

$$u_n = h\int_0^\tau\int_0^\infty (z+y)\exp(-h_1 y)F_n(\tau - s, z + y)\psi(s)\,dy\,ds + \qquad (2.3)$$

$$\int_0^\tau\int_0^\infty \Big[\Phi_n(\tau - s, |z-\xi|) - \Phi_n(\tau - s, z+\xi) + \int_0^\infty (z+\xi+y)\times$$

$$\exp(-h_1 y)F_n(\tau - s, z+\xi+y)\,dy\Big] f(s,\xi)\,d\xi\,ds + \int_0^\infty \Big[\Phi_n(\tau, |z -$$

$$\xi|) - \Phi_n(\tau, z+\xi) + \int_0^\infty (z+\xi+y)\exp(-h_1 y)F_n(\tau, z+\xi+y)\Big] b_1^2\varphi_1(\xi)\,d\xi$$

$$\Phi_n = \lim_{b_0\to 0}\Phi(\tau, z) = \frac{1}{2b_1\sqrt{\pi\tau}}\exp\left(-\frac{b_1^2 z^2}{4\tau}\right)$$

$$F_n = \lim_{b_0\to 0} F = \frac{b_1}{2\sqrt{\pi}\,\tau^{3/2}}\exp\left(-\frac{b_1^2 z^2}{4\tau}\right)$$

3. The temperature field. Assuming $f = \varphi_1 = \varphi_2 = 0$ in (2.2)

$$\psi(s) = \left(1 + \alpha_1\frac{\partial}{\partial s}\right)\varphi(s) = \frac{\alpha_0}{\tau_0}\left(1 + \frac{\alpha\tau_r}{\beta}\frac{\partial}{\partial s}\right)[sJ_-(s) - (s-\tau_0)J_- s - \tau_0)]$$

and replacing u by t, we obtain after elementary manipulations

$$t(\tau, z) = \frac{\alpha_0 h \exp(-k_1 b_0 z)}{\tau_0 (h_1 + k_1 b_0)} [\Phi_1(\tau, z) J_-(\tau - b_0 z) - \tag{3.1}$$

$$\Phi_1(\tau - \tau_0, z) J_-(\tau - \tau_0 - b_0 z)] + \frac{\alpha_0 \alpha_1 h k_1^2 b_0 \exp(-k_1 b_0 z)}{\tau_0 (h_1 + k_1 b_0)} \times$$

$$[\Phi_2(\tau, z) J_-(\tau - b_0 z) - \Phi_2(\tau - \tau_0, z) J_-(\tau - \tau_0 - b_0 z)] +$$

$$k_1 b_0 h \frac{\alpha_0}{\tau_0} [\Phi_3(\tau, z) J_-(\tau - b_0 z) - \Phi_3(\tau - \tau_0, z) J_-(\tau - \tau_0 - z b_0)] \equiv$$

$$F_2(\tau, z) J_-(\tau - b_0 z) - F_2(\tau - \tau_0, z) J_-(\tau - \tau_0 - b_0 z) \equiv$$

$$(I - T_\tau^{\tau_0}) [F_2(\tau, z) J_-(\tau - b_0 z)]$$

Here

$$\Phi_1(\tau, z) = \tau - b_0 z - \frac{1 - \alpha_1 d}{d} + \frac{1 - \alpha_1 d}{d} \exp[-d(\tau - b_0 z)]$$

$$\Phi_2(\tau, z) = z(\tau - b_0 z) + \frac{\tau - (2b_0 + \beta_1) z}{h_1 + k_1 b_0} - \frac{2(b_0 + \beta_1)}{(h_1 + k_1 b_0)^2} +$$

$$\exp[-d(\tau - b_0 z)] \left[\frac{\tau + \beta_1 z}{h_1 + k_1 b_0} + \frac{2(b_0 + \beta_1)}{(h_1 + k_1 b_0)^2} \right]$$

$$\Phi_3(\tau, z) = \int_0^{\tau_1} \int_0^{v_1} (z + y) \exp[-h_1 y - k_1(\tau - s - \beta_1 y)] \left\{ \left[1 - \alpha_1 k_1 - \right. \right.$$

$$\left. \frac{2\alpha_1 (\tau - s - \beta_1 y)}{(\tau - s - \beta_1 y)^2 - b_0^2 (z + y)^2} \right] \frac{I_1 (k_1 \sqrt{(\tau - s - \beta_1 y)^2 - b_0^2 (z + y)^2})}{\sqrt{(\tau - s - \beta_1 y)^2 - b_0^2 (z + y)^2}} +$$

$$\frac{k_1 \alpha_1 (\tau - s - \beta_1 y)}{(\tau - s - \beta_1 y)^2 - b_0^2 (z + y)^2} \times$$

$$\left. I_0 (k_1 \sqrt{(\tau - s - \beta_1 y)^2 - b_0^2 (z + y)^2}) \right\} s \, dy \, ds$$

$$\alpha_1 = \beta^{-1} \alpha \tau_r, \quad d = (b_0 + \beta_1)^{-1} (h_1 + k_1 b_0), \quad h_1 = \alpha h, \ \beta_1 = \alpha_1 h$$

$$\tau_1 = \tau - b_0 z, \quad v_1 = (b_0 + \beta_1)^{-1} (\tau - b_0 z - s)$$

The function $t(z, \tau)$ defined by (3.1) for $\alpha = \beta = 1$ describes the desired temperature field in the elastic half-space $z \geqslant 0$.

If the temperature or heat flux is specified on the boundary of the elastic half-space, then the temperature field has the form (3.1), where the functions

$$F_3(\tau, z) = \lim_{\substack{\beta \to \infty \\ h \to \infty}} F_2(\tau, z) |_{\alpha=1} = \frac{\alpha_0}{\tau_0} \left[(\tau - b_0 z) \exp(-k_1 b_0 z) + \right.$$

$$\left. k_1 b_0 z \int_{b_0 z}^{\tau} (\tau - \xi) \exp(-k_1 \xi) \frac{I_1 (k_1 \sqrt{\xi^2 - b_0^2 z^2})}{\sqrt{\xi^2 - b_0^2 z^2}} d\xi \right]$$

$$F_4(\tau, z) = \lim_{\alpha \to 0} F_2(\tau, z) |_{h=-1} = \frac{\alpha_0}{b_0 \tau_0} \times$$

$$\int_0^{\tau_1} s \exp(-k_1 (\tau - s)) I_0 (k_1 \sqrt{(\tau - s)^2 - b_0^2 z^2}) ds$$

must, respectively, replace $F_2(\tau, z)$.

The case of a jump change in the temperature on the boundary of an elastic half-space can be obtained from (3.1) with $\tau_0 \to 0$. Thus, (3.1) includes all the boundary conditions occurring most frequently in practice. Let us note that for $b_0 \to 0$ we obtain the corresponding parabolic (usual) temperature fields, and for $b_1 = 0$ the pure wave temperature fields satisfying conditions specified on the boundary.

4. The stress field. In (2.2) let us set

$$\varphi_1 = \varphi_2 = \psi = 0,\ \alpha = 1,\ b_0{}^2 = \frac{1}{c^2},\ b_1^2 = 0,\ f(s,\xi) = -a\frac{\partial^2 t(s,\xi)}{\partial s^2},\ u = \sigma_z$$

Then letting $h \to \infty$, $\beta \to \infty$, we obtain that the stress field in an elastic half-space is described by the functions (all the shear stresses equal zero)

$$\sigma_z(\tau, z) = \frac{ac}{2}\int_0^{\tau}\int_0^{\infty}\left[J_-\left(\tau - s - \frac{z+\xi}{c}\right) - J_-\left(\tau - s - \frac{|z-\xi|}{c}\right)\right] \times \tag{4.1}$$

$$\frac{\partial^2 t}{\partial s^2}d\xi\, ds = \frac{ac}{2}\left[\int_0^{c\tau - z}\frac{\partial t}{\partial s}\bigg|_{s=\tau-z+\xi,c} d\xi J_-\left(\tau - \frac{z}{c}\right) - \right.$$

$$\left.\int_z^{c\tau+z}\frac{\partial t}{\partial s}\bigg|_{s=\tau+z-\xi/c} d\xi - \int_0^z J_-\left(\tau - \frac{z-\xi}{c}\right)\frac{\partial t}{\partial s}\bigg|_{s=\tau-z-\xi/c} d\xi\right] =$$

$$\frac{ac}{2}(1 - T_\tau{}^{\tau_0})\left[J_-\left(\tau - \frac{z}{c}\right)\int_0^{c\tau-z} F_5\left(\tau - \frac{z+\xi}{c}, \xi\right)d\xi - \right.$$

$$\left.\int_z^{c\tau+z} F_5\left(\tau + \frac{z-\xi}{c}, \xi\right)d\xi - \int_0^z J_-\left(\tau - \frac{z-\xi}{c}\right)F_5\left(\tau - \frac{z-\xi}{c}, \xi\right)d\xi\right]$$

$$\sigma_x = \sigma_y = \frac{\mu}{1-\mu}\sigma_z - \frac{E\alpha_\tau}{1-\mu}t(\tau, z)$$

Here

$$F_5 = \frac{\alpha_0}{\tau_0}J_-(s - b_0\xi)\left[\frac{h\exp(-k_1 b_0\xi)}{h_1 + k_1 b_0}\Phi_4(s,\xi) + \right.$$

$$\left.\frac{\alpha_1 h k_1{}^2 b_0}{h_1 + k_1 b_0}\exp(-k_1 b_0 \xi)\Phi_5(s,\xi) + k_1 b_0 h \Phi_6(s,\xi)\right]$$

$$\Phi_4(s,\xi) = 1 - (1 - \alpha_1 d)\exp[-d(s - b_0\xi)]$$

$$\Phi_5(s,\xi) = \xi + \frac{1}{h_1 + k_1 b_0}\left\{1 - \exp[-d(s - b_0\xi)] - \right.$$

$$\left.\frac{s + \beta_1\xi}{b_0 + \beta_1}\exp[-d(s - b_0\xi)]\right\}$$

$$\Phi_6(s,\xi) = \int_0^{s_1}\left\{\left[1 - \alpha_1 k_1 + \frac{\alpha_1 k_1{}^2 b_0(s - \eta + \beta_1\xi)}{4(b_0 + \beta_1)}\right]\frac{k_1(s - \eta + \beta_1\xi)}{2(b_0 + \beta_1)^2} \times \right.$$

$$\left.\exp\left[-h_1\frac{s - \eta - b_0\xi}{b_0 + \beta_1} - k_1 b_0\frac{s - \eta + \beta_1\xi}{b_0 + \beta_1}\right]\right\}\eta d\eta + \int_0^{s_1}\eta\int_0^{v_2}(\xi + y) \times$$

$$\exp[-h_1 y - k_1(s - \eta - \beta_1 y)]\left\{\psi_1(s - \eta - \beta_1 y, \xi + y) I_0 \times \right.$$

$$(k_1\sqrt{(s - \eta - \beta_1 y)^2 + b_0{}^2(\xi + y)^2}) + \psi_2(s - \eta - \beta_1 y, \xi + y) \times$$

$$\left.\frac{I_1(k_1\sqrt{(s - \eta - \beta_1 y)^2 - b_0{}^2(\xi + y)^2})}{\sqrt{(s - \eta - \beta_1 y)^2 - b_0{}^2(\xi + y)^2}}\right\}dy d\eta$$

$$\psi_1(x,y) = \frac{k_1}{x^2 - b_0{}^2y^2}\left(x - 2\alpha_1 k_1 x - \frac{b_0{}^2y^2 + 3x^2}{x^2 - b_0{}^2y^2}\alpha_1\right)$$

$$\psi_2(x, y) = k_1(\alpha_1 k_1 - 1) + \frac{4\alpha_1 k_1 x - 2x + \alpha_1 k_1^2 x^2}{x^2 - b_0^2 y^2} + 2\,\frac{3\alpha_1 x^2 + \alpha_1 b_0^2 y^2}{(x^2 - b_0^2 y^2)^2}$$

$$s_1 = s - b_0\xi, \quad v_2 = (b_0 + \beta_1)^{-1}(s - \eta - b_0\xi)$$

where $t(\tau, z)$ is defined by (3.1).

The desired stress field in the half-space has the structure (3.1) for $\alpha = \beta = 1$, i.e.

$$\sigma_z = (I - T_\tau^{\tau_0})\left[F_6(\tau, z)\, J_-\left(\tau - \frac{z}{c}\right) + F_7(\tau, z)\, J_-(\tau - b_0 z)\right] \qquad (4.2)$$

and analogously for σ_x and σ_y.

Thus, the stress field in an elastic half-space $z \geqslant 0$ is obtained by the superposition of four kinds of waves: a heat wave with velocity $b_0 = 1/w_r$, a sound wave with velocity c, and the same waves but retarded by τ_0. In contrast to the parabolic case, the stress field is hence continuous.

Let us consider in greater detail the case when a thermal impact with a finite rate of change of the temperature is realized on the boundary of an elastic half-space. In this case the wave function F_5 is

$$F_5(s, \xi) = \frac{\alpha_0}{\tau_0} J_-(s - b_0\xi)\Big[\exp(-k_1 b_0 \xi) + \qquad (4.3)$$

$$k_1 b_0 \xi \int_{b_0\xi}^{s} \exp(-k_1 z)\, \frac{I_1(k_1\sqrt{z^2 - b_0^2\xi^2})}{\sqrt{z^2 - b_0^2\xi^2}}\Big]$$

Substituting (4.3) into (4.1), we obtain after evident manipulations that the stresses in the elastic half-space are described by the functions

$$\sigma_z = \frac{a\alpha_0}{b_1^2\tau_0^2}(I - T_\tau^{\tau_0})\Big\{J_-\left(\tau - \frac{z}{c}\right)\left[\exp\left(-k\left(\tau - \frac{z}{c}\right)\right) - 1\right] + \qquad (4.4)$$

$$J_-(\tau - b_0 z)\left[\exp(-k_1 b_0 z) - \exp(-k_1 b_0 z - k(\tau - b_0 z))\right] +$$

$$k_1 b_0 z J_-(\tau - b_0 z)\int_{b_0 z}^{\tau} \exp(-k_1\xi)\, \times$$

$$\frac{I_1(k_1\sqrt{\xi^2 - b_0^2 z^2})}{\sqrt{\xi^2 - b_0^2 z^2}}(1 - \exp(-k(\tau - \xi)))\, d\xi\Big\}$$

$$\sigma_x = \sigma_y = \frac{\mu}{1-\mu}\sigma_z - \frac{E\alpha_\tau}{1-\mu}t, \qquad k = \frac{b_1^2 c^2}{b_0^2 c^2 - 1} > 0$$

The quantity t is determined by (3.1), where F_2 is repalced by $F_3(\tau, z)$, and all the shear stresses are zero.

The following corollaries can be obtained from (4.4).

(1). If the temperature field is a pure wave one $(b_1 = 0)$, then

$$\sigma_z = \frac{a\alpha_0 c^2}{\tau_0(b_0^2c^2 - 1)}(I - T_\tau^{\tau_0})\left[(\tau - b_0 z)J_-(\tau - b_0 z) - \left(\tau - \frac{z}{c}\right)J_-\left(\tau - \frac{z}{c}\right)\right]$$

i.e. the stress field is linear in both time and the space variable.

(2). If a jump thermal impact is realized on the boundary of the ealstic half-space, i.e. $\tau_0 \to 0$, then

$$\sigma_z = \frac{a\alpha_0 c^2}{b_0^2 c^2 - 1}\Big\{\Big[\exp(-k_1 b_0 z - k(\tau - b_0 z)) + k_1 b_0 z\, \times \qquad (4.5)$$

$$\int_{b_0 z}^{\tau} \exp(-k\tau - k_1\xi + k\xi)\frac{I_1(k_1\sqrt{\xi^2 - b_0^2 z^2})}{\sqrt{\xi^2 - b_0^2 z^2}}\, d\xi\Big] \times$$

$$J_-(\tau - b_0 z) - J_-\left(\tau - \frac{z}{c}\right)\exp\left[-k\left(\tau - \frac{z}{c}\right)\right]\Big]\Big\}$$

(3). If the velocities of the thermal and elastic wave motions agree ($b_0 = 1/c$), then

$$\sigma_z = \frac{a\alpha_0}{\tau_0 b_1^2}(I - T_{\tau_0})\Big\{J_-\left(\tau - \frac{z}{c}\right)\Big[\exp\left(-\frac{1}{2} b_1^2 cz\right) -$$

$$1 + \frac{1}{2} b_1^2 cz \int_{z/c}^{\tau} \exp\left(-\frac{1}{2} b_1^2 c^2 \xi\right) \frac{I_1(^1/_2 b_1^2 c^2 \sqrt{\xi^2 - c^{-2} z^2})}{\sqrt{\xi^2 - c^{-2} z^2}}\, d\xi\Big]\Big\}$$

(4). If the temperature field is a pure wave field and the velocities of the thermal and elastic waves agree ($b_1 = 0,\ b_0 = 1/c$), then the stress field is linear in the space variable

$$\sigma_z = \frac{1}{2}\frac{ca\alpha_0}{\tau_0} z\left[J_-\left(\tau - \tau_0 - \frac{z}{c}\right) - J_-\left(\tau - \frac{z}{c}\right)\right]$$

and exists during the time $\tau \in (z/c,\ z/c + \tau_0)$. In the case of a thermal impact ($\tau_0 \to 0$), the stress σ_z acts at a concentrated time (instantaneously)

$$\sigma_z = -\,^1/_2 ca\alpha_0 z \delta(\tau - z/c)$$

(5). For $b_0 \to 0$ we obtain the case of the parabolic temperature field considered in [2] from (4.4).

If the heat flux on the boundary of an elastic half-space varies linearly, then the stress field is

$$\sigma_z = \frac{ac\alpha_0}{2b_0\tau_0}(I - T_{\tau_0})\Big\{J_-\left(\tau - \frac{z}{c}\right)\Big[\int_0^{\tau_2}\int_{b_0\xi}^{\xi_1} \exp(-k\xi) I_0(q)\, ds d\xi - \quad (4.6)$$

$$\int_0^{\tau_3}\int_{b_0\xi}^{\xi_2} \exp(-ks) I_0(q)\, ds d\xi\Big] + J_-(\tau - b_0 z)\Big[\int_z^{\tau_4}\int_{b_0\xi}^{\xi_3} \exp(-k_1 s) \times$$

$$I_0(q)\, ds d\xi - \int_z^{\tau_2}\int_{b_0\xi}^{\xi_1} \exp(-k_1 s) I_0(q)\, ds d\xi\Big]\Big\}$$

$$q = k_1\sqrt{s^2 - b_0^2\xi^2},\quad \tau_2 = \frac{c\tau - z}{cb_0 - 1},\quad \tau_3 = \frac{c\tau - z}{cb_0 + 1},\quad \tau_4 = \frac{c\tau + z}{cb_0 + 1}$$

$$\xi_1 = c^{-1}(c\tau - z + \xi),\ \xi_2 = c^{-1}(c\tau - z - \xi),\ \xi_3 = \tau + c^{-1}(z - \xi)$$

Evidently corollaries analogous to those obtained from (4.4) can be obtained from (4.6).

Graphs of the dependence of the stress $\bar{\sigma}_z = A^{-1}\sigma_z$ ($A = b_1^{-2} a\,\alpha_0$) on the time τ in

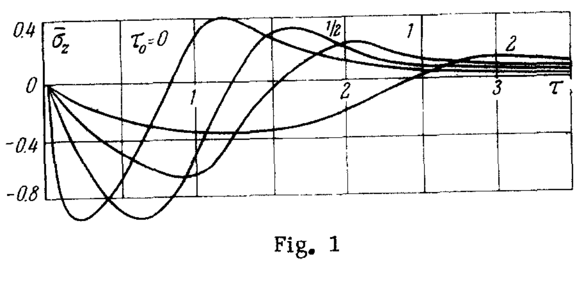

Fig. 1

the section $\xi = 1$ ($\xi = b_0 z$) for different heating times τ_0 have been constructed for a steel half-space by means of (4.4), (4.5).

It is seen from Fig. 1 that the maximum stress diminishes rapidly as τ_0 increases, and for $\tau_0 = 2$ this maximum is around 43% of its value at $\tau_0 = 0$ (instantaneous heating). Thus, the maximum dynamic stress is reduced 57% for a 2 sec heating duration. This indicates that taking account of the finite velocity of heat propagation, the rise in stress due to dynamic effects generally has no practical value.

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ON THE EXISTENCE OF A FIELD OF STRESS RATES IN A HARDENING ELASTIC-PLASTIC MEDIUM

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The boundary value problem for the stress rates and rates of change fields in the quasi-static motion of a volume V of an elastic-plastic medium [1] consists of finding the pairs $\sigma_{ij}^{\cdot}$, $\varepsilon_{ij}^{\cdot}$ related by the governing equations of an appropriate model; here the $\sigma_{ij}^{\cdot}$ should be statically admissible, i.e. should satisfy the equations and boundary conditions

$$\dot{\sigma}_{ij,\,j} = -\dot{X}_i, \qquad \dot{\sigma}_{ij} n_j |_{S_P} = \dot{p}_i \tag{0.1}$$

and $\varepsilon_{ij}^{\cdot}$ should be kinematically admissible, i.e. $2\varepsilon_{ij}^{\cdot} = v_{i,\,j} + v_{j,\,i}$, where

$$v_i |_{S_u} = \dot{u}_{i0} \tag{0.2}$$

Here S_p and S_u are nonintersecting parts of the boundary of the volume V, $X_i^{\cdot}$, $p_i^{\cdot}$, $u_{i0}^{\cdot}$ are specified functions. The question of the existence of a solution of this problem reduces to the question of the functional